Applied Engineering Mathematics

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Preface

Engineering mathematics including numerical methods and application is the essential part of key problem-solving skills for engineers and scientists. Modern engineering design and process modelling require both mathematical analysis and computer simulations. Vast literature exists on engineering mathematics, mathematical modelling and numerical methods. The topics in engineering mathematics are very diverse and the syllabus of mathematics itself is evolving. Therefore, there is a decision to select the topics and limit the number of chapters so that the book remains concise and yet comprehensive enough to include all the important mathematical methods and popular numerical methods.

This book endeavors to strike a balance between mathematical and numerical coverage of a wide range of mathematical methods and numerical techniques. It strives to provide an introduction, especially for undergraduates and graduates, to engineering mathematics and its applications. Topics include advanced calculus, ordinary differential equations, partial differential equations, vector and tensor analysis, calculus of variations, integral equations, the finite difference method, the finite volume method, the finite element method, reaction-diffusion system, and probability and statistics. The book also emphasizes the application of important mathematical methods with dozens of worked examples. The applied topics include elasticity, harmonic motion, chaos, kinematics, pattern formation and hypothesis testing. The book can serve as a textbook in engineering mathematics, mathematical modelling, and scientific computing.

Xin-She Yang
Cambridge, 2007

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Xin-She Yang

About the Author

Xin-She Yang received his D.Phil in applied mathematics from the University of Oxford. He is currently a research fellow at the University of Cambridge. Dr Yang has published extensively in international journals, book chapters, and conference proceedings. His research interests include asymptotic analysis, bioinspired algorithms, combustion, computational engineering, engineering optimization, solar eclipses, scientific programming and pattern formation. He is also the author of a book entitled: "An Introduction to Computational Engineering with Matlab", published in 2006 by Cambridge International Science Publishing Ltd.

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Chapter 1

Calculus

The preliminary requirements for this book are the pre-calculus foundation mathematics. We assume that the readers are familiar with these preliminaries, and readers can refer to any book that is dedicated to these topics. Therefore, we will only review some of the basic concepts of differentiation and integration.

1.1 Differentiations

1.1.1 Definition

For a known function or a curve $y = f(x)$ as shown in Figure 1.1, the slope or the gradient of the curve at the point $P(x, y)$ is defined as

$$\frac{dy}{dx} \equiv \frac{df(x)}{dx} \equiv f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}, \tag{1.1}$$

on the condition that there exists such a limit at P.

This gradient or limit is the first derivative of the function $f(x)$ at P. If the limit does not exist at a point P, then we say that the function is non-differentiable at P. By convention, the limit of the infinitesimal change Δx is denoted as the differential dx. Thus, the above definition can also be written as

$$dy = df = \frac{df(x)}{dx}dx = f'(x)dx, \tag{1.2}$$

which can be used to calculate the change in dy caused by the small change of dx. The primed notation $'$ and standard notation $\frac{d}{dx}$ can be used interchangeably, and the choice is purely out of convenience.

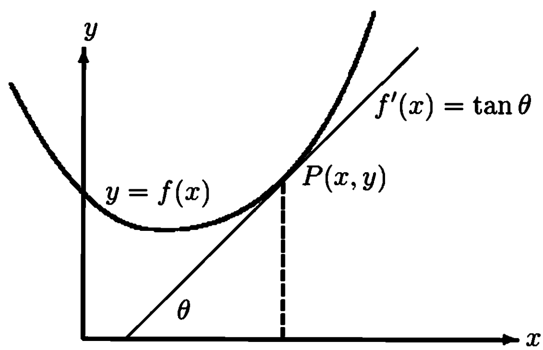

Figure 1.1: Gradient of a curve

The second derivative of $f(x)$ is defined as the gradient of $f'(x)$, or

$$\frac{d^2y}{dx^2} \equiv f''(x) = \frac{df(x)}{dx}. \tag{1.3}$$

The higher derivatives can be defined in a similar manner. Thus,

$$\frac{d^3y}{dx^3} \equiv f'''(x) = \frac{df''(x)}{dx}, \quad ..., \quad \frac{d^ny}{dx^n} \equiv f^{(n)} = \frac{df^{(n-1)}}{dx}. \tag{1.4}$$

1.1.2 Differentiation Rules

If a more complicated function $f(x)$ can be written as a product of two simpler functions $u(x)$ and $v(x)$, we can derive a differentiation rule using the definition from the first principles. Using

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{u(x+\Delta x)v(x+\Delta x)-u(x)v(x)}{\Delta x},$$

and subtracting and adding $-u(x+\Delta x)v(x)+u(x+\Delta x)v(x)$ [$=0$] terms, we have

$$\frac{df}{dx}=\frac{d[u(x)v(x)]}{dx}$$

$$=\lim_{\Delta x\to 0}[u(x+\Delta x)\frac{v(x+\Delta x)-v(x)}{\Delta x}+v(x)\frac{u(x+\Delta x)-u(x)}{\Delta x}]$$

$$=u(x)\frac{dv}{dx}+\frac{du}{dx}v(x), \tag{1.5}$$

which can be written in a contract form using primed notations

$$f'(x)=(uv)'=u'v+uv'. \tag{1.6}$$

If we differentiate this equation again and again, we can generalize this rule, we finally get the Leibnitz's Theorem for differentiations

$$\frac{d^n(uv)}{dx^n}=u^{(n)}v+nu^{(n-1)}v'+...+\begin{pmatrix} n \\ r \end{pmatrix}u^{(n-r)}v^{(r)}$$

$$+...+uv^{(n)}, \tag{1.7}$$

where the coefficients are the same as the binomial coefficients

$${}^nC_r \equiv \begin{pmatrix} n \\ r \end{pmatrix}=\frac{n!}{r!(n-r)!}. \tag{1.8}$$

If a function $f(x)$ [for example, $f(x)=e^{x^n}$] can be written as a function of another function $g(x)$, or $f(x)=f[g(x)]$ [for example, $f(x)=e^{g(x)}$ and $g(x)=x^n$], then we have

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta f}{\Delta g}\frac{\Delta g}{\Delta x}, \tag{1.9}$$

which leads to the following chain rule

$$f'(x) = \frac{df}{dg}\frac{dg}{dx}, \tag{1.10}$$

or

$$\{f[g(x)]\}' = f'[g(x)] \cdot g'(x). \tag{1.11}$$

In our example, we have $f'(x) = (e^{x^n})' = e^{x^n} n x^{n-1}$.

If one use $1/v$ instead of v in the equation (1.6) and $(1/v)' = -v'/v^2$, we have the following differentiation rule for quotients:

$$(\frac{u}{v})' = \frac{u'v - uv'}{v^2}. \tag{1.12}$$

□ ***Example 1.1:*** *The derivative of $f(x) = \sin(x)e^{-\cos(x)}$ can be obtained using the combination of the above differentiation rules.*

$$f'(x) = [\sin(x)]'e^{-\cos(x)} + \sin(x)[e^{-\cos(x)}]'$$
$$= \cos(x)e^{-\cos(x)} + \sin(x)e^{-\cos(x)}[-\cos(x)]'$$
$$= \cos(x)e^{-\cos(x)} + \sin^2(x)e^{-\cos(x)}.$$

□

The derivatives of various functions are listed in Table 1.1.

1.1.3 Implicit Differentiation

The above differentiation rules still apply in the case when there is no simple explicit function form $y = f(x)$ as a function of x only. For example, $y + \sin(x)\exp(y) = 0$. In this case, we can differentiate the equation term by term with respect to x so that we can obtain the derivative dy/dx which is in general a function of both x and y.

□ ***Example 1.2:*** *Find the derivative $\frac{dy}{dx}$ if $y^2 + \sin(x)e^y = \cos(x)$. Differentiating term by term with respect to x, we have*

$$2y\frac{dy}{dx} + \cos(x)e^y + \sin(x)e^y\frac{dy}{dx} = -\sin(x),$$
$$\frac{dy}{dx} = -\frac{\cos(x)e^y + \sin(x)}{2y + \sin(x)e^y}.$$

□

Table 1.1: First Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^x	e^x
$a^x (a > 0)$	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$

1.2 Integrations

1.2.1 Definition

Integration can be viewed as the inverse of differentiation. The integration $F(x)$ of a function $f(x)$ satisfies

$$\frac{dF(x)}{dx} = f(x), \tag{1.13}$$

or

$$F(x) = \int_{x_0}^{x} f(\xi) d\xi, \tag{1.14}$$

where $f(x)$ is called the integrand, and the integration starts from x_0 (arbitrary) to x. In order to avoid any potential confusion, it is conventional to use a dummy variable (say, ξ) in the integrand. As we know, the geometrical meaning of the first derivative is the gradient of the function $f(x)$ at a point P, the