$A < B$
(A is less than B)

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based on notes for the Math Olympiad Program (MOP)

Version 1.0, last revised August 2, 1999

Please send all comments and corrections to the author at `kedlaya@math.mit.edu`. Thank you!

Introduction

These notes constitute a survey of the theory and practice of inequalities. While their intended audience is high-school students, primarily present and aspiring participants of the Math Olympiad Program (MOP), I hope they prove informative to a wider audience. In particular, those who experience inequalities via the Putnam competition, or via problem columns in such journals as *Crux Mathematicorum* or the *American Mathematical Monthly*, should find some benefit.

Having named high-school students as my target audience, I must now turn around and admit that I have not made any effort to keep calculus out of the exposition, for several reasons. First, in certain places, rewriting to avoid calculus would make the exposition a lot more awkward. Second, the calculus I invoke is for the most part pretty basic, mostly properties of the first and second derivative. Finally, it is my experience that many Olympiad participants have studied calculus anyway. In any case, I have clearly flagged uses of calculus in the text, and I've included a crash course in calculus (Chapter 4) to fill in the details.

By no means is this primer a substitute for an honest treatise on inequalities, such as the *magnum opus* of Hardy, Littlewood and Pólya [2] or its latter-day sequel [1], nor for a comprehensive catalog of problems in the area, for which we have Stanley Rabinowitz' series [4]. My aim, rather than to provide complete information, is to whet the reader's appetite for this beautiful and boundless subject.

Also note that I have given geometric inequalities short shrift, except to the extent that they can be written in an algebraic or trigonometric form. ADD REFERENCE.

Thanks to Paul Zeitz for his MOP 1995 notes, upon which these notes are ultimately based. (In particular, they are my source for symmetric sum notation.) Thanks also to the participants of the 1998 and 1999 MOPs for working through preliminary versions of these notes.

Caveat solver!

It seems easier to fool oneself by constructing a false proof of an inequality than of any other type of mathematical assertion. All it takes is one reversed inequality to turn an apparently correct proof into a wreck. The adage "if it seems too good to be true, it probably is" applies in full force.

To impress the gravity of this point upon the reader, we provide a little exercise in mathematical proofreading. Of the following X proofs, only Y are correct. Can you spot the fakes?

PUT IN THE EXAMPLES.

Chapter 1

Separable inequalities

This chapter covers what I call "separable" inequalities, those which can be put in the form

$$f(x_1) + \cdots + f(x_n) \geq c$$

for suitably constrained $x_1, \ldots, x_n$. For example, if one fixes the product, or the sum, of the variables, the AM-GM inequality takes this form, and in fact this will be our first example.

1.1 Smoothing, convexity and Jensen's inequality

The "smoothing principle" states that if you have a quantity of the form $f(x_1) + \cdots + f(x_n)$ which becomes smaller as you move two of the variables closer together (while preserving some constraint, e.g. the sum of the variables), then the quantity is minimized by making the variables all equal. This remark is best illustrated with an example: the famous arithmetic mean and geometric mean (AM-GM) inequality.

Theorem 1 (AM-GM). *Let $x_1, \ldots, x_n$ be positive real numbers. Then*

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n},$$

with equality if and only if $x_1 = \cdots = x_n$.

Proof. We will make a series of substitutions that preserve the left-hand side while strictly increasing the right-hand side. At the end, the x_i will all be equal and the left-hand side will equal the right-hand side; the desired inequality will follow at once. (Make sure that you understand this reasoning before proceeding!)

If the x_i are not already all equal to their arithmetic mean, which we call a for convenience, then there must exist two indices, say i and j, such that $x_i < a < x_j$. (If the x_i were all bigger than a, then so would be their arithmetic mean, which is impossible; similarly if they were all smaller than a.) We will replace the pair x_i and x_j by

$$x_i' = a, x_j' = x_i + x_j - a;$$

by design, x_i' and x_j' have the same sum as x_i and x_j, but since they are closer together, their product is larger. To be precise,

$$a(x_i + x_j - a) = x_i x_j + (x_j - a)(a - x_i) > x_i x_j$$

because $x_j - a$ and $a - x_i$ are positive numbers.

By this replacement, we increase the number of the x_i which are equal to a, preserving the left-hand side of the desired inequality by increasing the right-hand side. As noted initially, eventually this process ends when all of the x_i are equal to a, and the inequality becomes equality in that case. It follows that in all other cases, the inequality holds strictly. □

Note that we made sure that the replacement procedure terminates in a finite number of steps. If we had proceeded more naively, replacing a pair of x_i by *their* arithmetic meaon, we would get an infinite procedure, and then would have to show that the x_i were "converging" in a suitable sense. (They do converge, but making this precise would require some additional effort which our alternate procedure avoids.)

A strong generalization of this smoothing can be formulated for an arbitrary convex function. Recall that a set of points in the plane is said to be convex if the line segment joining any two points in the set lies entirely within the set. A function f defined on an interval (which may be open, closed or infinite on either end) is said to be *convex* if the set

$$\{(x, y) \in \mathbb{R}^2 : y \geq f(x)\}$$

is convex. We say f is *concave* if $-f$ is convex. (This terminology was standard at one time, but today most calculus textbooks use "concave up" and "concave down" for our "convex" and "concave". Others use the evocative sobriquets "holds water" and "spills water".)

A more enlightening way to state the definition might be that f is convex if for any $t \in [0, 1]$ and any x, y in the domain of f,

$$tf(x) + (1 - t)f(y) \geq f(tx + (1 - t)y).$$

If f is continuous, it suffices to check this for $t = 1/2$. Conversely, a convex function is automatically continuous except possibly at the endpoints of the interval on which it is defined.

DIAGRAM.

Theorem 2. *If f is a convex function, then the following statements hold:*

1. *If $a \leq b < c \leq d$, then $\frac{f(c)-f(a)}{c-a} \leq \frac{f(d)-f(b)}{d-b}$. (The slopes of secant lines through the graph of f increase with either endpoint.)*

2. *If f is differentiable everywhere, then its derivative (that is, the slope of the tangent line to the graph of f is an increasing function.)*

The utility of convexity for proving inequalities comes from two factors. The first factor is Jensen's inequality, which one may regard as a formal statement of the "smoothing principle" for convex functions.

Theorem 3 (Jensen). *Let f be a convex function on an interval I and let $w_1, \dots, w_n$ be nonnegative real numbers whose sum is 1. Then for all $x_1, \dots, x_n \in I$,*

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq f(w_1 x_1 + \cdots + w_n x_n).$$

Proof. An easy induction on n, the case $n = 2$ being the second definition above. □

The second factor is the ease with which convexity can be checked using calculus, namely via the second derivative test.

Theorem 4. *Let f be a twice-differentiable function on an open interval I. Then f is convex on I if and only if $f''(x) \geq 0$ for all $x \in I$.*

For example, the AM-GM inequality can be proved by noting that $f(x) = \log x$ is concave; its first derivative is $1/x$ and its second $-1/x^2$. In fact, one immediately deduces a weighted AM-GM inequality; as we will generalize AM-GM again later, we will not belabor this point.

We close this section by pointing out that separable inequalities sometimes concern functions which are not necessarily convex. Nonetheless one can prove something!

Example 5 (USA, 1998). *Let $a_0, \dots, a_n$ be numbers in the interval $(0, \pi/2)$ such that*

$$\tan(a_0 - \pi/4) + \tan(a_1 - \pi/4) + \cdots + \tan(a_n - \pi/4) \geq n - 1.$$

Prove that $\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$.

Solution. Let $x_i = \tan(a_i - \pi/4)$ and $y_i = \tan a_i = (1 + x_i)/(1 - x_i)$, so that $x_i \in (-1, 1)$. The claim would follow immediately from Jensen's inequality if only the function $f(x) = \log(1 + x)/(1 - x)$ were convex on the interval $(-1, 1)$, but alas, it isn't. It's concave on $(-1, 0]$ and convex on $[0, 1)$. So we have to fall back on the smoothing principle.

What happens if we try to replace x_i and x_j by two numbers that have the same sum but are closer together? The contribution of x_i and x_j to the left side of the desired inequality is

$$\frac{1 + x_i}{1 - x_i} \cdot \frac{1 + x_j}{1 - x_j} = 1 + \frac{2}{\frac{x_i x_j + 1}{x_i + x_j} - 1}.$$

The replacement in question will increase $x_i x_j$, and so will decrease the above quantity provided that $x_i + x_j > 0$. So all we need to show is that we can carry out the smoothing process so that every pair we smooth satisfies this restriction.

Obviously there is no problem if all of the x_i are positive, so we concentrate on the possibility of having $x_i < 0$. Fortunately, we can't have more than one negative x_i, since $x_0 + \cdots + x_n \geq n - 1$ and each x_i is less than 1. (So if two were negative, the sum would be at most the sum of the remaining $n - 1$ terms, which is less than $n - 1$.) Moreover, if $x_0 < 0$, we could not have $x_0 + x_j < 0$ for $j = 1, \dots, n$, else we would have the contradiction

$$x_0 + \cdots + x_n \leq (1 - n) x_0 < n - 1.$$

Thus $x_0+x_j>0$ for some j, and we can replace these two by their arithmetic mean. Now all of the x_i are positive and smoothing (or Jensen) may continue without further restrictions, yielding the desired inequality. □

Problems for Section 1.1

1. Make up a problem by taking a standard property of convex functions, and specializing to a particular function. The less evident it is where your problem came from, the better!

2. Given real numbers $x_1, \dots, x_n$, what is the minimum value of

$$|x-x_1|+\cdots+|x-x_n|?$$

3. (via Titu Andreescu) If f is a convex function and x_1, x_2, x_3 lie in its domain, then

$$\begin{aligned} f(x_1)+f(x_2)+f(x_3)+f\left(\frac{x_1+x_2+x_3}{3}\right) \\ \geq \frac{4}{3}\left[f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_2+x_3}{2}\right)\right. \\ \left.+f\left(\frac{x_3+x_1}{2}\right)\right]. \end{aligned}$$

4. (USAMO 1974/1) For $a, b, c>0$, prove that $a^a b^b c^c \geq (abc)^{(a+b+c)/3}$.

5. (India, 1995) Let $x_1, \dots, x_n$ be n positive numbers whose sum is 1. Prove that

$$\frac{x_1}{\sqrt{1-x_1}}+\cdots+\frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}.$$

6. Let A, B, C be the angles of a triangle. Prove that

   1. $\sin A+\sin B+\sin C \leq 3\sqrt{3}/2$;
   2. $\cos A+\cos B+\cos C \leq 3/2$;
   3. $\sin A/2 \sin B/2 \sin C/2 \leq 1/8$;
   4. $\cot A+\cot B+\cot C \geq \sqrt{3}$ (i.e. the Brocard angle is at most $\pi/6$).

   (Beware: not all of the requisite functions are convex everywhere!)

7. (Vietnam, 1998) Let $x_1, \dots, x_n$ $(n \geq 2)$ be positive numbers satisfying

$$\frac{1}{x_1+1998}+\frac{1}{x_2+1998}+\cdots+\frac{1}{x_n+1998}=\frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998.$$

(Again, beware of nonconvexity.)

8. Let $x_1, x_2, \ldots$ be a sequence of positive real numbers. If a_k and g_k are the arithmetic and geometric means, respectively, of $x_1, \ldots, x_k$, prove that

$$\frac{a_n^n}{g_k^n} \geq \frac{a_{n-1}^{n-1}}{g_k^{n-1}} \tag{1.1}$$

$$n(a_n - g_n) \geq (n-1)(a_{n-1} - g_{n-1}). \tag{1.2}$$

These strong versions of the AM-GM inequality are due to Rado [2, Theorem 60] and Popoviciu [3], respectively. (These are just a sample of the many ways the AM-GM inequality can be sharpened, as evidenced by [1].)

1.2 Unsmoothing and noninterior extrema

A convex function has no interior local maximum. (If it had an interior local maximum at x, then the secant line through $(x-\epsilon, f(x-\epsilon))$ and $(x+\epsilon, f(x+\epsilon))$ would lie under the curve at x, which cannot occur for a convex function.)

Even better, a function which is *linear* in a given variable, as the others are held fixed, attains no extrema in either direction except at its boundary values.

Problems for Section 1.2

1. (IMO 1974/5) Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

2. (Bulgaria, 1995) Let $n \geq 2$ and $0 \leq x_i \leq 1$ for all $i = 1, 2, \ldots, n$. Show that

$$(x_1 + x_2 + \cdots + x_n) - (x_1x_2 + x_2x_3 + \cdots + x_nx_1) \leq \left\lfloor \frac{n}{2} \right\rfloor,$$

and determine when there is equality.

1.3 Discrete smoothing

The notions of smoothing and convexity can also be applied to functions only defined on integers.

Example 6. *How should n balls be put into k boxes to minimize the number of pairs of balls which lie in the same box?*

Solution. In other words, we want to minimize $\sum_{i=1}^{k} \binom{n_i}{2}$ over sequences $n_1, \dots, n_k$ of nonnegative integers adding up to n.

If $n_i - n_j \geq 2$ for some i, j, then moving one ball from i to j decreases the number of pairs in the same box by

$$\binom{n_i}{2} - \binom{n_i - 1}{2} + \binom{n_j}{2} - \binom{n_j + 1}{2} = n_i - n_j - 1 > 0.$$

Thus we minimize the number of pairs by making sure no two boxes differ by more than one ball; one can easily check that the boxes must each contain $\lfloor n/k \rfloor$ or $\lceil n/k \rceil$ balls. □

Problems for Section 1.3

1. (Germany, 1995) Prove that for all integers k and n with $1 \leq k \leq 2n$,

$$\binom{2n+1}{k-1} + \binom{2n+1}{k+1} \geq 2 \cdot \frac{n+1}{n+2} \cdot \binom{2n+1}{k}.$$

2. (Arbelos) Let $a_1, a_2, \dots$ be a convex sequence of real numbers, which is to say $a_{k-1} + a_{k+1} \geq 2a_k$ for all $k \geq 2$. Prove that for all $n \geq 1$,

$$\frac{a_1 + a_3 + \cdots + a_{2n+1}}{n+1} \geq \frac{a_2 + a_4 + \cdots + a_{2n}}{n}.$$

3. (USAMO 1993/5) Let $a_0, a_1, a_2, \dots$ be a sequence of positive real numbers satisfying $a_{i-1}a_{i+1} \leq a_i^2$ for $i = 1, 2, 3, \dots$. (Such a sequence is said to be *log concave*.) Show that for each $n \geq 1$,

$$\frac{a_0 + \cdots + a_n}{n+1} \cdot \frac{a_1 + \cdots + a_{n-1}}{n-1} \geq \frac{a_0 + \cdots + a_{n-1}}{n} \cdot \frac{a_1 + \cdots + a_n}{n}.$$

4. (MOP 1997) Given a sequence $\{x_n\}_{n=0}^{\infty}$ with $x_n > 0$ for all $n \geq 0$, such that the sequence $\{a^n x_n\}_{n=0}^{\infty}$ is convex for all $a > 0$, show that the sequence $\{\log x_n\}_{n=0}^{\infty}$ is also convex.

5. How should the numbers $1, \dots, n$ be arranged in a circle to make the sum of the products of pairs of adjacent numbers as large as possible? As small as possible?

Chapter 2

Symmetric polynomial inequalities

This section is a basic guide to polynomial inequalities, which is to say, those inequalities which are in (or can be put into) the form $P(x_1, \dots, x_n) \geq 0$ with P a symmetric polynomial in the real (or sometimes nonnegative) variables $x_1, \dots, x_n$.

2.1 Introduction to symmetric polynomials

Many inequalities express some symmetric relationship between a collection of numbers. For this reason, it seems worthwhile to brush up on some classical facts about symmetric polynomials.

For arbitrary $x_1, \dots, x_n$, the coefficients $c_1, \dots, c_n$ of the polynomial

$$(t + x_1) \cdots (t + x_n) = t^n + c_1 t^{n-1} + \cdots + c_{n-1} t + c_n$$

are called the *elementary symmetric functions* of the x_i. (That is, c_k is the sum of the products of the x_i taken k at a time.) Sometimes it proves more convenient to work with the *symmetric averages*

$$d_i = \frac{c_i}{\binom{n}{i}}.$$

For notational convenience, we put $c_0 = d_0 = 1$ and $c_k = d_k = 0$ for $k > n$.

Two basic inequalities regarding symmetric functions are the following. (Note that the second follows from the first.)

Theorem 7 (Newton). *If $x_1, \dots, x_n$ are nonnegative, then*

$$d_i^2 \geq d_{i-1} d_{i+1} \qquad i = 1, \dots, n-1.$$

Theorem 8 (Maclaurin). *If $x_1, \dots, x_n$ are positive, then*

$$d_1 \geq d_2^{1/2} \geq \cdots \geq d_n^{1/n},$$

with equality if and only if $x_1 = \cdots = x_n$.

These inequalities and many others can be proved using the following trick.

Theorem 9. *Suppose the inequality*

$$f(d_1, \dots, d_k) \geq 0$$

holds for all real (resp. positive) $x_1, \dots, x_n$ *for some* $n \geq k$*. Then it also holds for all real (resp. positive)* $x_1, \dots, x_{n+1}$.

Proof. Let

$$P(t) = (t + x_1) \cdots (t + x_{n+1}) = \sum_{i=0}^{n+1} \binom{n+1}{i} d_i t^{n+1-i}$$

be the monic polynomial with roots $-x_1, \dots, -x_n$. Recall that between any two zeros of a differentiable function, there lies a zero of its derivative (Rolle's theorem). Thus the roots of $P'(t)$ are all real if the x_i are real, and all negative if the x_i are positive. Since

$$P'(t) = \sum_{i=0}^{n+1} (n+1-i) \binom{n+1}{i} d_i t^{n-i} = (n+1) \sum_{i=0}^{n} \binom{n}{i} d_i t^{n-i},$$

we have by assumption $f(d_1, \dots, d_k) \geq 0$. □

Incidentally, the same trick can also be used to prove certain polynomial identities.

Problems for Section 2.1

1. Prove that every symmetric polynomial in $x_1, \dots, x_n$ can be (uniquely) expressed as a polynomial in the elementary symmetric polynomials.

2. Prove Newton's and Maclaurin's inequalities.

3. Prove Newton's identities: if $s_i = x_1^i + \cdots + x_n^i$, then

   $$c_0 s_k + c_1 s_{k-1} + \cdots + c_{k-1} s_1 + k c_k = 0.$$

   (Hint: first consider the case $n = k$.)

4. (Hungary) Let $f(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + 1$ be a polynomial with non-negative real coefficients and n real roots. Prove that $f(x) \geq (1+x)^n$ for all $x \geq 0$.

5. (Gauss-??) Let $P(z)$ be a polynomial with complex coefficients. Prove that the (complex) zeroes of $P'(z)$ all lie in the convex hull of the zeroes of $P(z)$. Deduce that if S is a convex subset of the complex plane (e.g., the unit disk), and if $\Re f(d_1, \dots, d_k) \geq 0$ holds for all $x_1, \dots, x_n \in S$ for some $n \geq k$, then the same holds for $x_1, \dots, x_{n+1} \in S$.

6. Prove Descartes' Rule of Signs: let $P(x)$ be a polynomial with real coefficients written as $P(x) = \sum a_{k_i} x^{k_i}$, where all of the a_{k_i} are nonzero. Prove that the number of positive roots of $P(x)$, counting multiplicities, is equal to the number of sign changes (the number of i such that $a_{k_{i-1}} a_{k_i} < 0$) minus a nonnegative even integer. (For negative roots, apply the same criterion to $P(-x)$.)

2.2 The idiot's guide to homogeneous inequalities

Suppose one is given a homogeneous symmetric polynomial P and asked to prove that $P(x_1, \dots, x_n) \geq 0$. How should one proceed?

Our first step is purely formal, but may be psychologically helpful. We introduce the following notation:

$$\sum_{\text{sym}} Q(x_1, \dots, x_n) = \sum_{\sigma} Q(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where σ runs over all permutations of $1, \dots, n$ (for a total of $n!$ terms). For example, if $n = 3$, and we write x, y, z for x_1, x_2, x_3,

$$\sum_{\text{sym}} x^3 = 2x^3 + 2y^3 + 2z^3$$

$$\sum_{\text{sym}} x^2 y = x^2 y + y^2 z + z^2 x + x^2 z + y^2 x + z^2 y$$

$$\sum_{\text{sym}} xyz = 6xyz.$$

Using symmetric sum notation can help prevent errors, particularly when one begins with rational functions whose denominators must first be cleared. Of course, it is always a good algebra check to make sure that equality continues to hold when it's supposed to.

In passing, we note that other types of symmetric sums can be useful when the inequalities in question do not have complete symmetry, most notably cyclic summation

$$\sum_{\text{cyclic}} x^2 y = x^2 y + y^2 z + z^2 x.$$

However, it is probably a bad idea to mix, say, cyclic and symmetric sums in the same calculation!

Next, we attempt to "bunch" the terms of P into expressions which are nonnegative for a simple reason. For example,

$$\sum_{\text{sym}} (x^3 - xyz) \geq 0$$

by the AM-GM inequality, while

$$\sum_{\text{sym}} (x^2 y^2 - x^2 yz) \geq 0$$

by a slightly jazzed-up but no more sophisticated argument: we have $x^2y^2 + x^2z^2 \geq x^2yz$ again by AM-GM.

We can formalize what we are doing using the notion of majorization. If $s = (s_1, \dots, s_n)$ and $t = (t_1, \dots, t_n)$ are two nonincreasing sequences, we say that s *majorizes* t if $s_1 + \cdots + s_n = t_1 + \cdots + t_n$ and $s_1 + \cdots + s_i \geq t_1 + \cdots + t_i$ for $i = 1, \dots, n$.

Theorem 10 ("Bunching"). *If s and t are sequences of nonnegative reals such that s majorizes t, then*

$$\sum_{\text{sym}} x_1^{s_1} \cdots x_n^{s_n} \geq \sum_{\text{sym}} x_1^{t_1} \cdots x_n^{t_n}.$$

Proof. One first shows that if s majorizes t, then there exist nonnegative constants k_σ, as σ ranges over the permutations of $\{1, \ldots, n\}$, whose sum is 1 and such that

$$\sum_{\sigma} k_\sigma (s_{\sigma_1}, \ldots, s_{\sigma_n}) = (t_1, \ldots, t_n)$$

(and conversely). Then apply weighted AM-GM as follows:

$$\begin{aligned}
\sum_{\sigma} x_1^{s_{\sigma(n)}} \cdots x_n^{s_{\sigma(n)}} &= \sum_{\sigma,\tau} k_\tau x_1^{s_{\sigma(\tau(1))}} \cdots x_n^{s_{\sigma(\tau(n))}} \\
&\geq \sum_{\sigma} x_1^{\sum_\tau k_\tau s_{\sigma(\tau(1))}} \cdots x_n^{\sum_\tau k_\tau s_{\sigma(\tau(n))}} \\
&= \sum_{\sigma} x_1^{t_{\sigma(1)}} \cdots x_n^{t_{\sigma(n)}}.
\end{aligned}$$

□

If the indices in the above proof are too bewildering, here's an example to illustrate what's going on: for $s = (5, 2, 1)$ and $t = (3, 3, 2)$, we have

$$(3, 3, 2) = (5, 2, 1)+$$

and so BLAH.

Example 11 (USA, 1997). *Prove that, for all positive real numbers a, b, c,*

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \leq \frac{1}{abc}.$$

Solution. Clearing denominators and multiplying by 2, we have

$$\sum_{\text{sym}} (a^3 + b^3 + abc)(b^3 + c^3 + abc)abc \leq 2(a^3 + b^3 + abc)(b^3 + c^3 + abc)(c^3 + a^3 + abc),$$

which simplifies to

$$\sum_{\text{sym}} a^7bc + 3a^4b^4c + 4a^5b^2c^2 + a^3b^3c^3 \leq \sum_{\text{sym}} a^3b^3c^3 + 2a^6b^3 + 3a^4b^4c + 2a^5b^2c^2 + a^7bc,$$

and in turn to

$$\sum_{\text{sym}} 2a^6b^3 - 2a^5b^2c^2 \geq 0,$$

which holds by bunching. □

In this case we were fortunate that after slogging through the algebra, the resulting symmetric polynomial inequality was quite straightforward. Alas, there are cases where bunching will not suffice, but for these we have the beautiful inequality of Schur.

Note the extra equality condition in Schur's inequality; this is a much stronger result than AM-GM, and so cannot follow from a direct AM-GM argument. In general, one can avoid false leads by remembering that all of the steps in the proof of a given inequality must have equality conditions at least as permissive as those of the desired result!

Theorem 12 (Schur)**.** *Let x, y, z be nonnegative real numbers. Then for any $r > 0$,*

$$x^r(x-y)(x-z) + y^r(y-z)(y-x) + z^r(z-x)(z-y) \geq 0.$$

Equality holds if and only if $x = y = z$, or if two of x, y, z are equal and the third is zero.

Proof. Since the inequality is symmetric in the three variables, we may assume without loss of generality that $x \geq y \geq z$. Then the given inequality may be rewritten as

$$(x-y)[x^r(x-z) - y^r(y-z)] + z^r(x-z)(y-z) \geq 0,$$

and every term on the left-hand side is clearly nonnegative. □

Keep an eye out for the trick we just used: creatively rearranging a polynomial into the sum of products of obviously nonnegative terms.

The most commonly used case of Schur's inequality is $r = 1$, which, depending on your notation, can be written

$$3d_1^3 + d_3 \geq 4d_1d_2 \quad \text{or} \quad \sum_{\text{sym}} x^3 - 2x^2y + xyz \geq 0.$$

If Schur is invoked with no mention r, you should assume $r = 1$.

Example 13. *(Japan, 1997)*

(Japan, 1997) Let a, b, c be positive real numbers, Prove that

$$\frac{(b+c-a)^2}{(b+c)^2+a^2} + \frac{(c+a-b)^2}{(c+a)^2+b^2} + \frac{(a+b-c)^2}{(a+b)^2+c^2} \geq \frac{3}{5}.$$

Solution. We first simplify slightly, and introduce symmetric sum notation:

$$\sum_{\text{sym}} \frac{2ab+2ac}{a^2+b^2+c^2+2bc} \leq \frac{24}{5}.$$

Writing $s = a^2 + b^2 + c^2$, and clearing denominators, this becomes

$$5s^2 \sum_{\text{sym}} ab + 10s \sum_{\text{sym}} a^2bc + 20 \sum_{\text{sym}} a^3b^2c \leq 6s^3 + 6s^2 \sum_{\text{sym}} ab + 12s \sum_{\text{sym}} a^2bc + 48a^2b^2c^2$$

which simplifies a bit to

$$6s^3 + s^2 \sum_{\text{sym}} ab + 2s \sum_{\text{sym}} a^2bc + 8 \sum_{\text{sym}} a^2b^2c^2 \geq 10s \sum_{\text{sym}} a^2bc + 20 \sum_{\text{sym}} a^3b^2c.$$

Now we multiply out the powers of s:

$$\sum_{\text{sym}} 3a^6 + 2a^5b - 2a^4b^2 + 3a^4bc + 2a^3b^3 - 12a^3b^2c + 4a^2b^2c^2 \geq 0.$$

The trouble with proving this by AM-GM alone is the $a^2b^2c^2$ with a *positive* coefficient, since it is the term with the most evenly distributed exponents. We save face using Schur's inequality (multiplied by $4abc$:)

$$\sum_{\text{sym}} 4a^4bc - 8a^3b^2c + 4a^2b^2c^2 \geq 0,$$

which reduces our claim to

$$\sum_{\text{sym}} 3a^6 + 2a^5b - 2a^4b^2 - a^4bc + 2a^3b^3 - 4a^3b^2c \geq 0.$$

Fortunately, this is a sum of four expressions which are nonnegative by weighted AM-GM:

$$\begin{aligned}
0 &\leq 2\sum_{\text{sym}}(2a^6 + b^6)/3 - a^4b^2 \\
0 &\leq \sum_{\text{sym}}(4a^6 + b^6 + c^6)/6 - a^4bc \\
0 &\leq 2\sum_{\text{sym}}(2a^3b^3 + c^3a^3)/3 - a^3b^2c \\
0 &\leq 2\sum_{\text{sym}}(2a^5b + a^5c + ab^5 + ac^5)/6 - a^3b^2c.
\end{aligned}$$

Equality holds in each case of AM-GM, and in Schur, if and only if $a = b = c$. □

Problems for Section 2.2

1. Suppose r is an even integer. Show that Schur's inequality still holds when x, y, z are allowed to be arbitrary real numbers (not necessarily positive).

2. (Iran, 1996) Prove the following inequality for positive real numbers x, y, z:

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4}.$$

3. The author's solution to the USAMO 1997 problem also uses bunching, but in a subtler way. Can you find it? (Hint: try replacing each summand on the left with one that factors.)

4. (MOP 1998)

5. Prove that for $x, y, z > 0$,

$$\frac{x}{(x+y)(x+z)} + \frac{y}{(y+z)(y+x)} + \frac{z}{(z+x)(z+y)} \leq \frac{9}{4(x+y+z)}.$$

2.3 Variations: inhomogeneity and constraints

One can complicate the picture from the previous section in two ways: one can make the polynomials inhomogeneous, or one can add additional constraints. In many cases, one can reduce to a homogeneous, unconstrained inequality by creative manipulations or substitutions; we illustrate this process here.

Example 14 (IMO 1995/2). *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution. We eliminate both the nonhomogeneity and the constraint by instead proving that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2(abc)^{4/3}}.$$

This still doesn't look too appetizing; we'd prefer to have simpler denominators. So we make the additional substitution $a = 1/x, b = 1/y, c = 1/z$ $a = x/y$, $b = y/z$, $c = z/x$, in which case the inequality becomes

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}(xyz)^{1/3}. \qquad (2.1)$$

Now we may follow the paradigm: multiply out and bunch. We leave the details to the reader. (We will revisit this inequality several times later.) □

On the other hand, sometimes more complicated maneuvers are required, as in this offbeat example.

Example 15 (Vietnam, 1996). *Let a, b, c, d be four nonnegative real numbers satisfying the conditions*

$$2(ab + ac + ad + bc + bd + cd) + abc + abd + acd + bcd = 16.$$

Prove that

$$a + b + c + d \geq \frac{2}{3}(ab + ac + ad + bc + bd + cd)$$

and determine when equality holds.

Solution (by Sasha Schwartz). Adopting the notation from the previous section, we want to show that
$$3d_2 + d_3 = 4 \implies d_1 \geq d_2.$$
Assume on the contrary that we have $3d_2 + d_3 = 4$ but $d_1 < d_2$. By Schur's inequality plus Theorem 9, we have
$$3d_1^3 + d_3 \geq 4d_1 d_2.$$
Substituting $d_3 = 4 - 3d_2$ gives
$$3d_1^3 + 4 \geq (4d_1 + 3)d_2 > 4d_1^2 + 3d_1,$$
which when we collect and factor implies $(3d_1 - 4)(d_1^2 - 1) > 0$. However, on one hand $3d_1 - 4 < 3d_2 - 4 = -d_3 < 0$; on the other hand, by Maclaurin's inequality $d_1^2 \geq d_2 > d_1$, so $d_1 > 1$. Thus $(3d_1 - 4)(d_1^2 - 1)$ is negative, a contradiction. As for equality, we see it implies $(3d_1 - 4)(d_1^2 - 1) = 0$ as well as equality in Maclaurin and Schur, so $d_1 = d_2 = d_3 = 1$. □

Problems for Section 2.3

1. (IMO 1984/1) Prove that $0 \leq yz + zx + xy - 2xyz \leq 7/27$, where x, y and z are non-negative real numbers for which $x + y + z = 1$.

2. (Ireland, 1998) Let a, b, c be positive real numbers such that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq abc$. (In fact, the right hand side can be improved to $\sqrt{3}abc$.)

3. (Bulgaria, 1998) Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq \frac{1}{2+a} + \frac{1}{2+b} + \frac{1}{2+c}.$$

2.4 Substitutions, algebraic and trigonometric

Sometimes a problem can be simplified by making a suitable substition. In particular, this technique can often be used to simplify unwieldy constraints.

One particular substitution occurs frequently in problems of geometric origin. The condition that the numbers a, b, c are the sides of a triangle is equivalent to the constraints
$$a + b > c, \quad b + c > a, \quad c + a > b$$
coming from the triangle inequality. If we let $x = (b + c - a)/2, y = (c + a - b)/2, z = (a + b - c)/2$, then the constraints become $x, y, z > 0$, and the original variables are also easy to express:
$$a = y + z, \quad b = z + x, \quad c = x + y.$$

A more exotic substitution can be used when the constraint $a + b + c = abc$ is given. Put
$$\alpha = \arctan a, \quad \beta = \arctan b, \quad \gamma = \arctan c;$$

then $\alpha+\beta+\gamma$ is a multiple of π. (If a, b, c are positive, one can also write them as the cotangents of three angles summing to $\pi/2$.)

Problems for Section 2.4

1. (IMO 1983/6) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geq 0,$$

   and determine when equality occurs.

2. (Asian Pacific, 1996) Let a, b, c be the lengths of the sides of a triangle. Prove that

$$\sqrt{a+b-c}+\sqrt{b+c-a}+\sqrt{c+a-b} \leq \sqrt{a}+\sqrt{b}+\sqrt{c}.$$

3. (MOP, 1999) Let a, b, c be lengths of the the sides of a triangle. Prove that

$$a^3+b^3+c^3-3abc \geq 2\max\{|a-b|^3, |b-c|^3, |c-a|^3\}.$$

4. (Arbelos) Prove that if a, b, c are the sides of a triangle and

$$2(ab^2+bc^2+ca^2) = a^2b+b^2c+c^2a+3abc,$$

   then the triangle is equilateral.

5. (Korea, 1998) For positive real numbers a, b, c with $a+b+c=abc$, show that

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2},$$

   and determine when equality occurs. (Try proving this by dehomogenizing and you'll appreciate the value of the trig substitution!)

Chapter 3

The toolbox

In principle, just about any inequality can be reduced to the basic principles we have outlined so far. This proves to be fairly inefficient in practice, since once spends a lot of time repeating the same reduction arguments. More convenient is to invoke as necessary some of the famous classical inequalities described in this chapter.

We only barely scratch the surface here of the known body of inequalities; already [2] provides further examples, and [1] more still.

3.1 Power means

The power means constitute a simultaneous generalization of the arithmetic and geometric means; the basic inequality governing them leads to a raft of new statements, and exposes a symmetry in the AM-GM inequality that would not otherwise be evident.

For any real number $r \neq 0$, and any positive reals $x_1, \dots, x_n$, we define the r-th power mean of the x_i as

$$M^r(x_1, \dots, x_n) = \left(\frac{x_1^r + \cdots + x_n^r}{n} \right)^{1/r}.$$

More generally, if $w_1, \dots, w_n$ are positive real numbers adding up to 1, we may define the weighted r-th power mean

$$M_w^r(x_1, \dots, x_n) = (w_1 x_1^r + \cdots + w_n x_n^r)^{1/r}.$$

Clearly this quantity is continuous as a function of r (keeping the x_i fixed), so it makes sense to define M^0 as

$$\begin{aligned} \lim_{r \to 0} M_w^r &= \lim_{r \to 0} \left(\frac{1}{r} \exp \log(w_1 x_1^r + \cdots + w_n x_n^r) \right) \\ &= \exp \left. \frac{d}{dr} \right|_{r=0} \log(w_1 x_1^r + \cdots + w_n x_n^r) \\ &= \exp \frac{w_1 \log x_1 + \cdots + w_n \log x_n}{w_1 + \cdots + w_n} = x_1^{w_1} \cdots x_n^{w_n} \end{aligned}$$

or none other than the weighted geometric mean of the x_i.

Theorem 16 (Power mean inequality)**.** *If $r > s$, then*

$$M_w^r(x_1, \ldots, x_n) \geq M_w^s(x_1, \ldots, x_n)$$

with equality if and only if $x_1 = \cdots = x_n$.

Proof. Everything will follow from the convexity of the function $f(x) = x^r$ for $r \geq 1$ (its second derivative is $r(r-1)x^{r-2}$), but we have to be a bit careful with signs. Also, we'll assume neither r nor s is nonzero, as these cases can be deduced by taking limits.

First suppose $r > s > 0$. Then Jensen's inequality for the convex function $f(x) = x^{r/s}$ applied to $x_1^s, \ldots, x_n^s$ gives

$$w_1 x_1^r + \cdots + w_n x_n^r \geq (w_1 x_1^s + \cdots + w_n x_n^s)^{r/s}$$

and taking the $1/r$-th power of both sides yields the desired inequality.

Now suppose $0 > r > s$. Then $f(x) = x^{r/s}$ is concave, so Jensen's inequality is reversed; however, taking $1/r$-th powers reverses the inequality again.

Finally, in the case $r > 0 > s$, $f(x) = x^{r/s}$ is again convex, and taking $1/r$-th powers preserves the inequality. (Or this case can be deduced from the previous ones by comparing both power means to the geometric mean.) □

Several of the power means have specific names. Of course $r = 1$ yields the arithmetic mean, and we defined $r = 0$ as the geometric mean. The case $r = -1$ is known as the *harmonic mean*, and the case $r = 2$ as the *quadratic mean* or *root mean square*.

Problems for Section 3.1

1. (Russia, 1995) Prove that for $x, y > 0$,

$$\frac{x}{x^4 + y^2} + \frac{y}{y^4 + x^2} \leq \frac{1}{xy}.$$

2. (Romania, 1996) Let $x_1, \ldots, x_{n+1}$ be positive real numbers such that $x_1 + \cdots + x_n = x_{n+1}$. Prove that

$$\sum_{i=1}^n \sqrt{x_i(x_{n+1} - x_i)} \leq \sqrt{\sum_{i=1}^n x_{n+1}(x_{n+1} - x_i)}.$$

3. (Poland, 1995) For a fixed integer $n \geq 1$ compute the minimum value of the sum

$$x_1 + \frac{x_2^2}{2} + \cdots + \frac{x_n^n}{n}$$

given that $x_1, \ldots, x_n$ are positive numbers subject to the condition

$$\frac{1}{x_1} + \cdots + \frac{1}{x_n} = n.$$